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Unique Distinction of the Lorentz and the Anti-de Sitter Group in the Classical Kepler Problem*

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Abstract

In contrast to the familiar group-theoretical investigations of the Kepler problem, providing rather different symmetry structures for different types of state, the question is posed whether the problem does not exhibit uniquely distinguished dynamic invariance symmetries, which may be considered characteristic for the Newtonian interaction, independent of the special type of orbit. Taking into account only one Laplace vector it is shown that one, and only one, such symmetry exists and that this is just that of the Lorentz group. One is led to consider a distinguished orthonormal dynamical frame of the system, providing an extension of the Lorentz invariance symmetry to that of the anti-de Sitter group. Finally, some remarks are made concerning the relativistic formulation and solution of the Kepler problem.

1. *Introduction*

During the last 15 years, much work has been done on the group-theoretical discussion of the Kepler problem either classical or quantum theoretical. [As representative for the state of affairs and for further references the two books of M. J. Englefield (1972) and of G. Wybourne (1974) may be cited.] This interest originates in the fact that the dynamical symmetries of the Kepler problem represent an ideal example for similar symmetries in more complicated and less understood physical systems, as might be seen, for instance, from A. O. Barut's book (1972) on dynamical groups and generalized symmetries with applications in atomic and particle physics. The intention of the present paper differs from this traditional line.

Since the contributions of L. Hulthén (1933), who realized an idea of O. Klein, and of V. Fock (1935) and V. Bargmann (1936) it is known that the three-dimensional kinematical Lie algebra $so(3)$ defined by the Poisson brackets of the angular momentum can be expanded to a six-dimensional dynamical Lie algebra

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so(4), or *e*(3), or *so*(3, 1) by adding the second integral vector of Laplace,[1] according to whether the energy is negative, or zero, or positive, and with appropriate factors added to the Laplace vector. From the usual more computational viewpoint this did not seem disturbing [cf., e.g., p. 37 of A. Decoster's thesis (1970)[2]]. However, if one takes notice of the essential connection of Newton's potential with the existence of the Laplace integral vector, the dynamical symmetry represented by this vector should be considered characteristic of the Newtonian potential, so that it appears extremely dissatisfying that it is not *one* distinguished "hidden" symmetry that manifests itself as a dynamical invariance group[3] of the Kepler problem, independent of the state of the system, or with another word, *adiabatically invariant*, because every orbit might be transformed adiabatically into any other possible orbit. Thus, the object of the present paper is to answer the following three questions:

1. Does there exist an adiabatically invariant dynamical invariance symmetry induced by the angular momentum vector **l** and a Laplace vector, and if so, is there only one, and then, which one?
2. In case (1) has a positive answer, does there exist a higher dynamical invariance group of which the former one is a subgroup?
3. What may be the physical conclusions?

These three questions define the subdivision of the following investigation. The answers to question (1) and (2) are as follows:

1. Question (1) amounts to a simple partial differential equation problem with one and only one solution, namely, a Laplace vector with modulus $l := |\mathbf{l}|$. Either of the two Laplace vectors may be taken, and furthermore, any linear combination of the two vectors which has the modulus l, too. The Lie algebra thus obtained is the *Lorentz algebra so*(3, 1)!
2. The fact that the two Laplace vectors (normed to modulus l) and the angular momentum vector represent a *group-theoretically distinguished orthonormal dynamical frame* suggests considering the Poisson brackets of these two Laplace vectors as well. Then it is seen immediately that only l must be added as a tenth constant variable to provide a closed algebra of Poisson bracket relations. The invariance algebra thus obtained is the *anti-de Sitter algebra so*(3, 2).

We shall confine the following discussion to the standpoint of classical

1 This vector is also called the eccentricity vector or Runge–Lenz vector if the modulus is taken equal to the numerical eccentricity.—As to its intricate history, attention should be drawn to a profound investigation of O. Volk (1976).

2 This paper contains a very meritorious compilation of the earlier literature on the invariance groups of the hydrogen atom.

3 As usual, "invariance" is meant with respect to the Hamiltonian, i.e., as constancy in time; "symmetry," on the other hand, is a neutral word, here, for either group or Lie algebra.

mechanics with Poisson brackets defined as

$$[f,g] := \sum_k \left(\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k}\right)$$

2. *The Problem of the Laplace Vector's Constant Factor*

The five integrals of the relative motion of the Kepler problem are the Hamiltonian of the relative motion

$$H := \mathbf{p}^2 | (2m) - \alpha^2 | r \tag{2.1}$$

the angular momentum of the relative motion

$$\mathbf{l} := \mathbf{r} \times \mathbf{p} \tag{2.2}$$

and, for instance, the dimensionless second Laplace vector

$$\mathbf{A} := \frac{\mathbf{r}\cdot\mathbf{p}}{m\alpha^2}\,\mathbf{p} - \left(\frac{\mathbf{p}^2}{m\alpha^2} - \frac{1}{r}\right)\mathbf{r} \tag{2.3}$$

This vector points to the aphelion and represents only one additional integral because of $\mathbf{A}\cdot\mathbf{l} = 0$ and

$$\mathbf{A}^2 = e^2 := 1 + 2Hl^2/(m\alpha^4) \tag{2.4}$$

where e is the numerical *eccentricity*. W. Pauli (1926) derived the following Poisson bracket relations:

$$[l_i, l_j] = -e_{ijk} l_k \tag{2.5a}$$

$$[A_i, l_j] = -e_{ijk} A_k \tag{2.5b}$$

$$[A_i, A_j] = \frac{2H}{m\alpha^4}\, e_{ijk} l_k \tag{2.5c}$$

where e_{ijk} means the three-dimensional permutation tensor and the summation convention is used. For $H = 0$ this is a realization of the Lie algebra $e(3)$. Because H commutes with $\mathbf{l}$ and $\mathbf{A}$, it is quite obvious to define for $H \neq 0$ two new vectors according to

$$\mathbf{n}^{\pm} := \left(\frac{m\alpha^4}{\pm 2H}\right)^{1/2} \mathbf{A} \qquad \text{for } H \gtrless 0 \tag{2.6}$$

so that instead of (2.5a)–(2.5c)

$$[l_i, l_j] = -e_{ijk} l_k \tag{2.7a}$$

$$[n_i^{\pm}, l_j] = -e_{ijk} n_k^{\pm} \tag{2.7b}$$

$$[n_i^{\pm}, n_j^{\pm}] = \pm e_{ijk} l_k \qquad \text{for } H \gtrless 0 \tag{2.7c}$$

thus rendering Lie algebras for $H \neq 0$, too, namely the algebra $so(3,1)$ if $H > 0$ (unbound states), and $so(4)$ if $H < 0$ (bound states), as stated already by L. Hulthén (1933).

Instead of $\mathbf{n}^{\pm}$ we consider now the most general second Laplace vector

$$\mathbf{k} := f(H, l^2)\mathbf{A} \tag{2.8}$$

for this equation (2.7b) remains valid whereas equation (2.7c) must be replaced by

$$\begin{aligned}[k_i, k_j] &= f^2 \cdot \frac{2H}{m\alpha^4} e_{ijk} l_k + f \frac{\partial f(H, l^2)}{\partial (l^2)} \cdot 2e_{ijk} [\mathbf{A}^2 l_k - (\mathbf{l} \cdot \mathbf{A}) A_k] \\ &= \frac{\partial}{\partial (l^2)} (e^2 f^2)|_H \cdot e_{ijk} l_k \end{aligned} \tag{2.9}$$

if equations (2.5b) and (2.5c), and $\mathbf{A}^2 = e^2$ and $\mathbf{l} \cdot \mathbf{A} = 0$ are taken into account. Therefore, the demand that the Poisson brackets of the l_i and k_i form a Lie algebra amounts to

$$e^2 f^2 = \epsilon l^2 + g(H) \tag{2.10}$$

where, without loss of generality, the structure constant ϵ (which must be dimensionless for physical reasons) might be taken as

$$\epsilon = \pm 1 \tag{2.10a}$$

Then, $g(H)$ should have the dimension l^2, and therefore, because of dimensional reasons,

$$g(H) = \delta \cdot m\alpha^4/(2H) \tag{2.10b}$$

(δ : pure number). If $H \neq 0$, the usual vectors $\mathbf{n}^{\pm}$, as defined by equation (2.6), are obtained with

$$\epsilon = \delta = \pm 1 \qquad \text{for } H \gtrless 0 \tag{2.11}$$

However, an adiabatically invariant real $\mathbf{k}$ is obtained *if and only if*

$$\epsilon = 1, \qquad \delta = 0 \tag{2.12}$$

so that

$$[l_i, l_j] = -e_{ijk} l_k \tag{2.13a}$$

$$[k_i, l_j] = -e_{ijk} k_k \tag{2.13b}$$

$$[k_i, k_j] = e_{ijk} l_k \tag{2.13c}$$

with

$$\mathbf{k} = \mathbf{k}_{(2)} := -l\mathbf{A}/|\mathbf{A}| \tag{2.14}$$

for instance. As mentioned above, equations (2.13a)-(2.13c) realize the *Lorentz algebra* $so(3, 1)$.

From thus it follows immediately that with a first Laplace vector

$$\mathbf{k} = \mathbf{k}_{(1)} := \mathbf{k}_{(2)} \times \mathbf{l}/l = \mathbf{l} \times \mathbf{A}/|\mathbf{A}| \tag{2.15}$$

instead of the expression (2.14), equations (2.13a)-(2.13c) still remain valid; equation (2.13b) is easily proved, whilst the verification of equation (2.13c) is a little cumbersome. Indeed, any constant vector **k** of the form

$$\mathbf{k} = \cos\psi\, \mathbf{k}_{(1)} + \sin\psi\, \mathbf{k}_{(2)} \tag{2.16}$$

(ψ: real number) satisfies equations (2.13b) and (2.13c), as is easily verified with equation (2.13b) and equation (3.4) of the following section.

More generally and independent of Newton's law of attraction, *any, not necessarily constant, vector* **k** *orthogonal on* **l**, *and with modulus l, satisfies equations (2.13b) and (2.13c).* Indeed, any such vector might be written as

$$\mathbf{k} = f(l, p_r, r, t)\mathbf{r} - g(l, p_r, r, t)\mathbf{p} \tag{2.17}$$

(where p_r is the radial momentum) with f and g subjected to the condition $\mathbf{k}^2 = l^2$ only. However, any vector of the type (2.17) satisfies equation (2.13b). Thus, forming $[k_i, \mathbf{k}\cdot\mathbf{l}] \equiv 0$, and considering equation (2.13b), one obtains

$$[k_i, k_j]\, l_j = 0 \tag{2.18}$$

which amounts to

$$[k_i, k_j] = F(l, p_r, r, t)\cdot e_{ijk} l_k \tag{2.19}$$

so that

$$[k_i, \mathbf{k}^2] = 2F e_{ijk} k_j l_k \tag{2.20}$$

On the other hand, equation (2.13b) leads to

$$[k_i, l^2] = 2e_{ijk} k_j l_k \tag{2.21}$$

so that $F = 1$ because of $\mathbf{k}^2 = l^2$. QED.

Thus, we have arrived at the surprising result that the symmetry of *the Lorentz group is dynamically distinguished in more than one respect in the classical Kepler problem already.* Especially, it is hidden in the Kepler problem as a *uniquely distinguished dynamical invariance symmetry*! In accordance with the current suggestive name "boost" for the generator of a pure Lorentz transformation, one might call the vectors **k** *"boost vectors" of the relative motion*, in analogy with **l** being called the angular momentum of the relative motion. Remembering that particle systems in many respects behave like one single particle – e.g., gravitational n-body systems in a homogeneous gravitational field or H_2 molecules and He atoms in refraction experiments with molecular beams (Estermann and Stern, 1930) – we can look at a Kepler system as one single particle with constant *spin* $\mathbf{S} = \mathbf{l}$, which generates an internal rotation group of this "particle." We have learned now that the constant spin $\mathbf{S} = \mathbf{l}$ and a constant internal boost vector **k** must be considered as one constant unit, namely, a *Minkowskian spin tensor* $S_{\kappa\lambda} = -S_{\lambda\kappa}$,

$$(S_{\kappa\lambda}) := \begin{pmatrix} 0 & l_3 & -l_2 & k_1 \\ -l_3 & 0 & l_1 & k_2 \\ l_2 & -l_1 & 0 & k_3 \\ -k_1 & -k_2 & -k_3 & 0 \end{pmatrix} \tag{2.22}$$

generating an *internal Lorentz group* which originates from the Newtonian internal interaction of the "particle" called Kepler system. This internal realization of the Lorentz algebra has the remarkable property that both its Casimir operators vanish identically,

$$C_1 := \tfrac{1}{2} S_{\kappa\lambda} S^{\kappa\lambda} = \mathbf{l}^2 - \mathbf{k}^2 \equiv 0 \tag{2.23a}$$

$$C_2 := \tfrac{1}{8} e^{\kappa\lambda\mu\nu} S_{\kappa\lambda} S_{\mu\nu} = \mathbf{l} \cdot \mathbf{k} \equiv 0 \tag{2.23b}$$

where $e^{\kappa\lambda\mu\nu}$ is the four-dimensional permutation symbol, and summation runs from 1 to 4, and a diagonal metric $(g_{\kappa\lambda}) = (1, 1, 1, -1)$ is assumed.[4]

3. *Extension of the Dynamical Invariance Symmetry*

Evidently, the two boost vectors $\mathbf{k}_{(1)}$, $\mathbf{k}_{(2)}$, and the angular momentum $\mathbf{l}$ represent a distinguished orthonormal *dynamical frame.* Therefore, it is obvious to calculate also the Poisson brackets $[k_{(1)i}, k_{(2)j}]$.

Taking care of

$$[k_{(1)i}, l] = -k_{(2)i} \tag{3.1}$$

$$[k_{(2)i}, l] = k_{(1)i} \tag{3.2}$$

one obtains

$$[k_{(1)i}, k_{(2)j}] = -2l\delta_{ij} + (k_{(1)i}k_{(1)j} + k_{(2)i}k_{(2)j} + l_i l_j)/l \tag{3.3}$$

where the second term is equal to

$$l\mathbf{e}_i \cdot \mathbf{e}_j = l\delta_{ij}$$

(the $\mathbf{e}_i$ are the Cartesian basis vectors) so that

$$[k_{(1)i}, k_{(2)j}] = -l\delta_{ij} \tag{3.4}$$

This suggests the introduction of l as a tenth constant variable. Indeed, if we define the five-dimensional antimetric matrix

$$(S_{ab}) := \begin{pmatrix} 0 & l_3 & -l_2 & k_{(1)1} & k_{(2)1} \\ -l_3 & 0 & l_1 & k_{(1)2} & k_{(2)2} \\ l_2 & -l_1 & 0 & k_{(1)3} & k_{(2)3} \\ -k_{(1)1} & -k_{(1)2} & -k_{(1)3} & 0 & l \\ -k_{(2)1} & -k_{(2)2} & -k_{(2)3} & -l & 0 \end{pmatrix} \tag{3.5}$$

4 It should be noticed that $C_1 \equiv 0$ is not true with the usual vectors $\mathbf{n}^{\pm}$.

and the five-dimensional metric

$$(g_{ab}) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.6}$$

we see that

$$[S_{ab}, S_{cd}] = -g_{ac}S_{bd} - g_{bd}S_{ac} + g_{ad}S_{bc} + g_{bc}S_{ad} \tag{3.7}$$

According to a well-known formula[5] this means that the Poisson bracket relations of the S_{ab} realize the *anti-de Sitterian Lie algebra* $so(3, 2)$, thus establishing this symmetry as a still larger dynamical invariance symmetry. Thus, the Newtonian interaction even induces in a unique way the existence of a *constant five-dimensional spin tensor* S_{ab} generating an *internal anti-de Sitter group* of the "particle" called Kepler system. This S_{ab} realization of the $so(3, 2)$ has the remarkable property, too, that both its Casimir operators vanish identically:

$$C_1 := \tfrac{1}{2}S_{ab}S^{ab} = \mathbf{l}^2 - \mathbf{k}_{(1)}^2 - \mathbf{k}_{(2)}^2 + l^2 \equiv 0 \tag{3.8a}$$

$$C_2 := W_a W^a \equiv 0 \tag{3.8b}$$

because with

$$W^a := \tfrac{1}{8} e^{abcde} S_{bc} S_{de} \tag{3.9}$$

(e^{abcde}: five-dimensional permutation tensor) one has

$$(W^a) = \begin{pmatrix} \mathbf{W} \\ W^4 \\ W^5 \end{pmatrix} = \begin{pmatrix} l\mathbf{l} - \mathbf{k}_{(1)} \times \mathbf{k}_{(2)} \\ -\mathbf{l} \cdot \mathbf{k}_{(2)} \\ \mathbf{l} \cdot \mathbf{k}_{(1)} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix} \tag{3.10}$$

4. *Physical Conclusions*

The unique distinction of the fundamental Lorentz symmetry by the Newtonian interaction in the classical Kepler problem urges the conjecture that the opposite might be true, too, – namely, that *a correct Lorentz covariant formulation of the classical problem of two bodies distinguishes the Newtonian interaction in a unique way.*

At first sight, this seems to be a rather irreal conclusion, because a lot of papers were occupied with relativistic particle dynamics since A. Sommerfeld's treatment of the relativistic Kepler problem, and never any hint of such a dis-

[5] Cf., e.g., p. 187 in the book of R. Gilmore (1974). This formula was published for the first time by W. Pauli in his famous "Handbuch-Article" (1933), p. 180.

tinction was found. On the other hand, one should remember that relativistic particle dynamics never ceased to be a controversial field in classical physics – reaching up to so-called no-interaction theorems – as might be seen, for instance, from the preface of E. H. Kerner (1972) or from the book of E. C. G. Sudarshan and N. Mukunda (1974). Indeed, opinions not only diverge as to whether the equations of motions should be integrodifferential equations or Hamiltonian differential equations, but even the notion of Lorentz covariance is differently used by different authors. Furthermore, most investigations are mathematically rather complicated, so that concrete problems are seldom solved, or if they are, as for instance with H. P. Künzle (1974), then it is only for special cases. At this state of affairs the above-mentioned conjecture concerning the Newtonian potential might seem less irreal, and if there exist different possibilities for defining Lorentz covariance of particle dynamics, one might hope that a unique distinction of Newton's interaction could serve as a distinguishing criterion.

The Hamiltonian character of the investigation of Section 2 suggests that a correctly formulated relativistic Kepler problem should have Hamiltonian and manifestly[6] Lorentz covariant character so that the kinematical integrals

$$l_{ij} := x_i p_j - x_j p_i, \qquad i, j = 1, 2, 3 \tag{4.1a}$$

must be supplemented by a further kinematical integral three-vector

$$l_{i4} := x_i p_4 - x_4 p_i, \qquad i = 1, 2, 3 \tag{4.1b}$$

which, because of its orthogonality with the $l_i = \frac{1}{2} e_{ijk} l_{jk}$, represents only two new integrals, and which should become a Laplace vector in the nonrelativistic limit, thus revealing the true meaning of this mysterious vector. Therefore, a connection between the additional internal canonical variables p_4 and x_4, and the functions $f_{(1)}, g_{(1)}$, or $f_{(2)}, g_{(2)}$, defined by

$$k_{(a)i} = f_{(a)} x_i - g_{(a)} p_i, \qquad a = 1, 2 \tag{4.2}$$

should be expected, which means that the eight variables $\mathbf{x}, x_4, \mathbf{p}, p_4$ of the relative motion must be subjected to two constraints. In fact, as will be shown in a further paper, this program can be performed in a straightforward manner, because a suitably constructed covariant Hamiltonian formalism for two interacting particles necessarily includes two nonholonomic constraints that restrict the possible interaction so severely that with certain simplicity assumptions the Newton interaction is the only possible one. Indeed, essential features of this formalism are already contained in an interesting paper of P. M. Pearle's (1968)[7]; but the two constraints suggested by the Laplace vector and its intrinsic connection with the Lorentz group were not at P. M. Pearle's disposal, so that he could not treat bound states.

[6] "Manifestly" is meant here in a stronger sense than, for instance, in the book of E. C. G. Sudarshan and N. Mukunda (1974)!

[7] The author thanks Mr. G. Streicher for drawing his attention to this paper.

As to the result of Section 3, comparable conclusions cannot be drawn, because the physical meaning of the de Sitter group is understood much less. It should be noted, however, that the anti-de Sitter space possesses several interesting properties, as mentioned, for instance, by S. W. Hawking and G. F. R. Ellis (1973). Furthermore, it seems satisfying that we have not obtained the de Sitter group $SO(4, 1)$ instead of $SO(3, 2)$, because according to a remark of T. Philips' (1962), mentioned by F. Gürsey (1964), the group $SO(4, 1)$ may raise questions regarding the positive definiteness of the energy.

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